Non-equilibrium flow over a wavy wall

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A small-disturbance solution is obtained for the steady two-dimensional flow over a sinusoidal wall of an inviscid gas in vibrational or chemical non-equilibrium. The results are based on a single, linear, third-order partial differential equation, which plays the same role here as does the Prandtl–Glauert equation in equilibrium flow. The solution is valid throughout the range from subsonic to supersonic speeds and for all values of the rate parameter from equilibrium to frozen flow (in both of which limits it reduces to Ackeret's classical solution of the Prandtl–Glauert equation). The results illustrate in simple fashion some of the properties of non-equilibrium flow, such as the occurrence of pressure drag at subsonic speeds and the absence of the discontinuous phenomena that characterize the Prandtl–Glauert theory when the flow changes from subsonic to supersonic.

1. Introduction

This work was motivated by the desire to obtain an analytical solution—preferably a simple one—to some problem in the two-dimensional flow of a gas in vibrational or chemical non-equilibrium. It appeared at the outset that the most likely possibility for such a solution would be the small-disturbance problem of the steady flow over an infinite sinusoidal wall. Thirty years ago, Ackeret (1928) showed that this problem of the 'wavy wall' admits of a particularly simple solution when the gas is in thermodynamic equilibrium. As it turns out, interesting and instructive results can again be obtained in the non-equilibrium case, still on the basis of relatively simple mathematics.

The material that follows is a logical supplement to the work of Chu (1957), Wood & Kirkwood (1957a), Moore (1958), and Broer (1958), all of whom have studied the phenomena of wave propagation under non-equilibrium conditions. As a result of this work, the roles played by the so-called 'frozen' and 'equilibrium' speeds of sound in problems of wave propagation are now well understood. (The equilibrium speed of sound is the speed of sound calculated on the assumption that the vibrational and chemical states of the fluid change so as to maintain their equilibrium relationship with the other state variables at every instant; the frozen speed of sound is calculated on the assumption that vibrational and chemical states are fixed—that is, do not change at all.) It is clear, in particular, that for any finite reaction rate in the fluid the front of an infinitesimal wave propagates relative to the fluid at the frozen speed of sound, which is invariably the greater of the two. This is true no matter how large the reaction rate, so long as it is not infinite. Only when the rate becomes infinitely large does the wave-front velocity drop back to the lower equilibrium speed of sound. This discontinuous change in wave-front velocity in the limit of an infinite reaction rate is related to a reduction in the order of the governing differential equation with a resulting abrupt change in the identity of the characteristic surfaces. Despite this discontinuous change in wave-front velocity, however, the flow field is found to depend on the reaction rate in a continuous and uniform manner. This has been demonstrated by Chu (1957), who showed how a solution calculated for a large but finite reaction rate tends in the limit of infinite rate to the same solution as would be obtained by making the rate infinite at the outset. For the details the reader is referred to Chu's paper. We need only note here that, despite the discontinuous change in wave-front speeds, the flow calculated on the physically unrealistic basis of infinite rates (i.e. complete equilibrium) does have meaning with reference to a real flow with large but finite rates.

The foregoing, in brief, is the state of affairs in one-dimensional unsteady flow. A similar situation prevails in the two-dimensional steady case. Again there is a discontinuous change in the order of the governing differential equation, and hence in the characteristics of the equation, when the reaction rate becomes infinite. Despite this, Ackeret's solution for identically infinite rates (i.e. complete equilibrium) reappears as the natural limit of the non-equilibrium solution when the rate tends to infinity. This is true both in supersonic flow, where the characteristics are real and the problem is again essentially one in wave propagation, as well as in subsonic flow, where the characteristics are imaginary.

As in the note by Moore (1958), the results will be obtained by solution of a single third-order partial differential equation, which is here derived directly on the assumption of steady flow. This equation, which appears as equation (27), plays the same role in non-equilibrium theory as does the classical Prandtl–Glauert equation in equilibrium flow. It might therefore be useful as the basis for a theory of thin airfoils in a reactive fluid. The equation also provides one more example of a type of linear equation that arises in a number of problems in which there is a time lag between different state properties of the medium.

Subsequent to the completion of the present work, the author received a report by Gibson & Moore (1958) that does in fact consider the problem of the thin airfoil in supersonic flow on the basis of the same differential equation used here, derived by these writers via the corresponding acoustic equation. This was followed by a further paper by the same authors (Moore & Gibson, 1959) that also includes results for the wavy wall. The methods of the present paper are, however, formally different from those of Gibson & Moore.

The methods and results that follow can be applied equally well to vibrational non-equilibrium of a single-component gas or to chemical non-equilibrium of a mixture in which a single reaction occurs. Since understanding rather than application is the aim, no attempt is made to extend the treatment to multiple processes. All effects of viscosity, heat conduction, and diffusion are neglected.

2. Basic equations

We begin by setting down the equations of gas dynamics for three-dimensional time dependent flow. If ρ is the mass density and u_i ($i = 1, 2, 3$) are the velocity components, the continuity equation can be written

$$\frac{D\rho}{Dt} + \rho \frac{\partial u_j}{\partial x_j} = 0, \tag{1}$$

where the repeated dummy subscripts denote the summation convention and the substantial derivative is given as usual by $D(\;)/Dt \equiv \partial(\;)/\partial t + u_k \partial(\;)/\partial x_k$. If p is the pressure, the Eulerian equations of motion under the present assumptions are

$$\frac{Du_i}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial x_i} = 0 \quad (i = 1, 2, 3). \tag{2}$$

The adiabatic energy equation can be written in several equivalent forms. If h is the enthalpy per unit mass, two of these are

$$\frac{D}{Dt}\left(\frac{u^2}{2} + h\right) - \frac{1}{\rho}\frac{\partial p}{\partial t} = 0, \tag{3a}$$

$$\frac{Dh}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt} = 0. \tag{3b}$$

The equivalence of these equations can be shown with the aid of equation (2).

The above equations (1), (2) and (3) constitute five equations for the six unknowns ρ, u_i, p, and h. To complete the set we must introduce equations specifying the thermo-chemical properties of the gas.

For present purposes we consider the non-equilibrium thermo-chemical state to be described by the usual thermodynamic variables p, ρ, T, h, e, s, etc., plus an additional quantity q that specifies the vibrational or chemical state of the gas. (For vibrational non-equilibrium, q is taken as T_i, the internal vibrational temperature; for chemical non-equilibrium of a dissociating diatomic gas, it would be taken as α, the degree of dissociation.) For the assumed non-equilibrium situation, the specification of any three of these variables fixes the thermo-chemical state—that is, all of the other variables are then determined.

Following Broer (1958), we shall find it convenient to use p, ρ, and q as the primary state variables. The enthalpy h is thus assumed to be given by a state equation of the form

$$h = h(p, \rho, q). \tag{4a}$$

This may also be written in differential form as

$$dh = h_p dp + h_\rho d\rho + h_q dq, \tag{4b}$$

where the subscript indicates differentiation with respect to the noted variable, the other two state variables being held fixed.

The variable q is governed in a gas at rest by a rate equation which gives dq/dt as a function of the state of the gas and which can be written in the form

$$\frac{dq}{dt} = \frac{L(p, \rho, q)}{\theta}, \tag{5}$$

where L is a function of the thermo-chemical state and θ is a positive quantity which is related to the specific rate constant. (The smaller is θ, the faster is the reaction rate, and vice versa.) The quantity θ is also, in general, a function of the thermo-chemical state, but that need not concern us here. When the gas is in equilibrium, then dq/dt is zero, and equation (5) defines an equilibrium value of q—say $\bar{q}$—given by

$$L(p,\rho,\bar{q}) = 0, \tag{6a}$$

from which

$$\bar{q} = \bar{q}(p,\rho). \tag{6b}$$

There are, correspondingly, equilibrium values of the other state variables, such as (cf. equation (4*a*))

$$\bar{h} = h[p,\rho,\bar{q}(p,\rho)] = \bar{h}(p,\rho). \tag{7}$$

For the gas in motion, we assume that the rate equation following a fluid element has the same form as equation (5), that is

$$\frac{Dq}{Dt} = \frac{L(p,\rho,q)}{\theta}. \tag{8}$$

In general, Dq/Dt for the gas in motion has a finite value in regions between shock waves.

The case of infinitely fast reaction rates corresponds to letting $\theta \to 0$ in the rate equation. In this limit, since Dq/Dt is finite, equation (8) reduces simply to $L(p,\rho,q) = 0$. This means (cf. equation (6*a*)) that the state of the gas in this case is determined everywhere by the equilibrium relation $q = \bar{q}$. We thus see that the flow in the limit of infinitely fast reaction rates is the classical equilibrium flow. (For a more complete discussion of this limiting process, see Chu (1957).) The case of infinitely slow reaction rates is found by letting $\theta \to \infty$ in equation (8), which gives $q =$ constant. The resulting flow is the so-called frozen flow.

Equations (1), (2), (3), (4) and (8) now constitute a set of seven equations for the seven unknowns ρ, u_i, p, h, and q. They are thus sufficient for a solution of the problem.

If we wish to consider the entropy s of the non-equilibrium gas, this can be done by means of a state relation of the form

$$dh = T\,ds + \frac{1}{\rho}dp + Q\,dq, \tag{9}$$

where T is the absolute temperature and h is now considered as a function of s, p, and q. For chemical non-equilibrium, the quantity $Q \equiv (\partial h/\partial q)_{s,p}$ is related to the chemical potentials; for vibrational non-equilibrium it is equal to $c_i(T_i - T)\,dT_i/T_i$, where c_i is the vibrational specific heat (see, for example, Wood & Kirkwood (1957*b*)). For a system in equilibrium, it is known from chemical thermodynamics that h must be a minimum with respect to any virtual change in q for fixed s and p. The equilibrium value of $\bar{q} = \bar{q}(p,\rho)$ can therefore be found also from the equation

$$Q(p,\rho,\bar{q}) = 0. \tag{10}$$

The result must, of course, be the same as that found from equation (6*a*).

If equation (9) is applied following a fluid element, we have, in view of the energy equation (3*b*),

$$\frac{Ds}{Dt} = -\frac{Q}{T}\frac{Dq}{Dt}. \tag{11}$$

It is thus seen that for the limiting cases of equilibrium flow ($q = \bar{q}$ and hence $Q = 0$) and frozen flow ($Dq/Dt = 0$), the changes of state of a fluid element are isentropic. For any intermediate case ($Q \neq 0$ and $Dq/Dt \neq 0$), it follows from the second law of thermodynamics as applied to the present adiabatic situation that

$$\frac{Ds}{Dt} > 0,$$

that is, the flow process must be irreversible.

3. Linearized equations

Since our ultimate concern is with the small-disturbance flow over a wavy wall, we shall linearize the equations at this point. This step could be deferred until later, but it will simplify the manipulation of the equations to introduce it here. We shall also restrict ourselves to steady flow.

We assume as usual that the flow field is described by a perturbation on a uniform parallel flow with velocity U_∞ in the x_1 direction. If the perturbation quantities are denoted by primes, the velocity components are then given by $u_1 = U_\infty + u_1'$, $u_2 = u_2'$, $u_3 = u_3'$. The thermodynamic variables are given correspondingly by $p = p_\infty + p'$, $h = h_\infty + h'$, etc. The undisturbed stream is assumed to be in equilibrium—that is, $q_\infty = \bar{q}_\infty$.

The linearized form of the substantial derivative in the case of steady flow is $D(\ )/Dt = U_\infty \partial(\ )/\partial x_1$. With this expression the conservation equations (1), (2) and (3a) can be linearized at once to obtain

$$\rho_\infty \frac{\partial u_j'}{\partial x_j} + U_\infty \frac{\partial \rho'}{\partial x_1} = 0, \tag{12}$$

$$\rho_\infty U_\infty \frac{\partial u_i'}{\partial x_1} + \frac{\partial p'}{\partial x_i} = 0 \quad (i = 1, 2, 3), \tag{13}$$

$$U_\infty \frac{\partial u_1'}{\partial x_1} + \frac{\partial h'}{\partial x_1} = 0. \tag{14}$$

The linearized form of the differential state relation (4b) is

$$dh' = h_{p\infty} dp' + h_{\rho\infty} d\rho' + h_{q\infty} dq', \tag{15}$$

where the partial derivatives are now evaluated at the conditions of the undisturbed stream.

The linearization of the rate equation requires somewhat more attention. By expanding the function $L(p, \rho, q)$ about the free-stream condition, equation (8) can be approximated first by

$$U_\infty \frac{\partial q'}{\partial x_1} = \frac{L(p_\infty, \rho_\infty, q_\infty) + L_{p\infty} p' + L_{\rho\infty} \rho' + L_{q\infty} q'}{\theta_\infty + \theta'}.$$

Since the free-stream is in equilibrium, we have $L(p_\infty, \rho_\infty, q_\infty) = L(p_\infty, \rho_\infty, \bar{q}_\infty) = 0$ (cf. equation (6a)), and the equation can be simplified further to

$$U_\infty \frac{\partial q'}{\partial x_1} = \frac{1}{\theta_\infty}(L_{p\infty} p' + L_{\rho\infty} \rho' + L_{q\infty} q'). \tag{16}$$

We now introduce the concept of a fictitious, *local* equilibrium value of q given by $\bar{q}(p,\rho)$. This is the equilibrium value of q corresponding to the actual (non-equilibrium) values of p and ρ at a local point in the field. It is *not* the same as the value of q that would exist at that point if the entire field were in equilibrium—that is, in the limit of infinitely fast reaction rates. This fictitious equilibrium value $\bar{q}$ can be related to the actual values of p and ρ by equation (6*a*). Expanding the left-hand side of this equation about the free-stream condition and writing $\bar{q} = \bar{q}_\infty + \bar{q}' = q_\infty + \bar{q}'$ (where $\bar{q}_\infty = q_\infty$, as before, since the free stream is in equilibrium), equation (6*a*) can be linearized as

$$L_{p\infty} p' + L_{\rho\infty} \rho' + L_{q\infty} \bar{q}' = 0.$$

With the aid of this relation, equation (16) can be written finally as

$$U_\infty \frac{\partial q'}{\partial x_1} = \frac{\bar{q}' - q'}{\tau_\infty}, \tag{17}$$

where $\tau_\infty \equiv -\theta_\infty / L_{q\infty}$ is the relaxation time of the non-equilibrium process evaluated at free-stream conditions. Equation (17) could have been obtained by assuming a linearized rate equation at the outset on the plausible grounds that the flow over a slightly wavy wall could never be far out of equilibrium. The present approach has the advantage of showing that this simplified rate equation is a formal consequence of the other approximations in the linearized analysis.

The linearized rate equation (17) introduces the new unknown $\bar{q}'$ into the analysis. We must therefore add an additional equation. This is given by equation (6*b*), which can be written in linearized differential form as

$$d\bar{q}' = \bar{q}_{p\infty} dp' + \bar{q}_{\rho\infty} d\rho'. \tag{18}$$

Equations (12), (13), (14), (15), (17) and (18) now constitute eight linearized equations in eight unknowns.

The foregoing equations will now be combined to obtain a single equation with the velocity components as the unknowns. We begin by using the state equation (15) to rewrite the energy equation (14) as

$$U_\infty \frac{\partial u_1'}{\partial x_1} + h_{p\infty} \frac{\partial p'}{\partial x_1} + h_{\rho\infty} \frac{\partial \rho'}{\partial x_1} + h_{q\infty} \frac{\partial q'}{\partial x_1} = 0.$$

The derivatives $\partial \rho'/\partial x_1$ and $\partial p'/\partial x_1$ are next eliminated from this equation by means of the continuity equation (12) and the first of the Eulerian equations (13). The result can be written

$$U_\infty h_{q\infty} \frac{\partial q'}{\partial x_1} = (\rho_\infty h_{p\infty} - 1)\, U_\infty^2 \frac{\partial u_1'}{\partial x_1} + \rho_\infty h_{\rho\infty} \frac{\partial u_j'}{\partial x_j}. \tag{19}$$

Now, turning to the linearized rate equation (17), we differentiate this with respect to x_1 and write with the aid of equation (18)

$$\tau_\infty U_\infty \frac{\partial^2 q'}{\partial x_1^2} = \bar{q}_{p\infty} \frac{\partial p'}{\partial x_1} + \bar{q}_{\rho\infty} \frac{\partial \rho'}{\partial x_1} - \frac{\partial q'}{\partial x_1}.$$

This can be further rewritten with the aid of equations (12) and (13) as

$$\tau_\infty U_\infty \frac{\partial^2 q'}{\partial x_1^2} + \frac{\partial q'}{\partial x_1} + \rho_\infty U_\infty \bar{q}_{p\infty} \frac{\partial u_1'}{\partial x_1} + \frac{\rho_\infty}{U_\infty} \bar{q}_{\rho\infty} \frac{\partial u_j'}{\partial x_j} = 0. \tag{20}$$

All that need be done now to obtain an equation in the u_i is to differentiate equation (19) with respect to x_1 and use the result and the original equation (19) to eliminate the derivatives of q' from equation (20). This gives finally, after some rearrangement of terms,

$$\frac{h_{\rho\infty}U_\infty\tau_\infty}{h_{\rho\infty}+h_{q\infty}\bar{q}_{\rho\infty}}\frac{\partial}{\partial x_1}\left[\frac{h_{p\infty}-1/\rho_\infty}{h_{\rho\infty}}U_\infty^2\frac{\partial u_1'}{\partial x_1}+\frac{\partial u_j'}{\partial x_j}\right]+\left[\frac{h_{p\infty}+h_{q\infty}\bar{q}_{p\infty}-1/\rho_\infty}{h_{\rho\infty}+h_{q\infty}\bar{q}_{\rho\infty}}U_\infty^2\frac{\partial u_1'}{\partial x_1}+\frac{\partial u_j'}{\partial x_j}\right]=0. \tag{21}$$

As in the work of Broer (1958), the coefficients of the first terms inside the brackets can be related to the frozen and equilibrium speeds of sound. To this end we use the general thermodynamic relation

$$\left(\frac{\partial p}{\partial \rho}\right)_s = -\frac{(\partial h/\partial \rho)_p}{(\partial h/\partial p)_\rho - 1/\rho}. \tag{22}$$

The square of the frozen speed of sound a_f^2 is obtained by taking $\partial p/\partial \rho$ with both s and q fixed. We thus obtain, in view of the relation (22),

$$a_f^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_{s,q} = -\frac{(\partial h/\partial \rho)_{p,q}}{(\partial h/\partial p)_{\rho,q}-1/\rho} = -\frac{h_\rho}{h_p - 1/\rho}. \tag{23}$$

The square of the equilibrium speed of sound a_e^2 is obtained by taking $\partial p/\partial \rho$ with s fixed and equilibrium maintained. This last condition means that we hold $q = \bar{q}(p,\rho)$, or equivalently $h = h[p,\rho,\bar{q}(p,\rho)] = \bar{h}(p,\rho)$. We thus obtain from equation (22)

$$a_e^2 \equiv \left[\left(\frac{\partial p}{\partial \rho}\right)_s\right]_{h=\bar{h}} = -\frac{(\partial \bar{h}/\partial \rho)_p}{(\partial \bar{h}/\partial p)_\rho - 1/\rho} = -\frac{h_\rho + h_q\bar{q}_\rho}{h_p + h_q\bar{q}_p - 1/\rho}. \tag{24}$$

With the aid of these last two relations, equation (21) can be written finally as

$$K\frac{\partial}{\partial x_1}\left[(1-M_{f\infty}^2)\frac{\partial u_1'}{\partial x_1}+\frac{\partial u_2'}{\partial x_2}+\frac{\partial u_3'}{\partial x_3}\right]+\left[(1-M_{e\infty}^2)\frac{\partial u_1'}{\partial x_1}+\frac{\partial u_2'}{\partial x_2}+\frac{\partial u_3'}{\partial x_3}\right]=0, \tag{25}$$

where $M_{f\infty} = U_\infty/a_{f\infty}$, $M_{e\infty} = U_\infty/a_{e\infty}$, and

$$K = \frac{h_{\rho\infty}U_\infty\tau_\infty}{h_{\rho\infty}+h_{q\infty}\bar{q}_{\rho\infty}}. \tag{26}$$

Equation (25) is the desired single equation for the velocity components.

To show that a velocity potential may be introduced, we rewrite the three linearized Euler equations (13) as the single vector equation

$$\rho_\infty U_\infty \frac{\partial \mathbf{u}'}{\partial x_1} + \operatorname{grad} p' = 0,$$

and take the curl of this equation. This gives

$$\frac{\partial}{\partial x_1}\operatorname{curl}\mathbf{u}' = 0.$$

In view of the linearized relation $D(\ )/Dt = U_\infty \partial(\ )/\partial x_1$, this result means that the vorticity of a fluid element remains constant. Since we are concerned with perturbations on an initially uniform flow in which the vorticity is zero, it follows that the perturbed flow will remain irrotational. We may therefore introduce a perturbation potential ϕ such that $u'_i = \partial\phi/\partial x_i$. Equation (25) then becomes

$$K\frac{\partial}{\partial x_1}\left[(1-M_{f\infty}^2)\frac{\partial^2\phi}{\partial x_1^2}+\frac{\partial^2\phi}{\partial x_2^2}+\frac{\partial^2\phi}{\partial x_3^2}\right]+\left[(1-M_{e\infty}^2)\frac{\partial^2\phi}{\partial x_1^2}+\frac{\partial^2\phi}{\partial x_2^2}+\frac{\partial^2\phi}{\partial x_3^2}\right]=0. \quad (27)$$

The foregoing is the governing differential equation for non-equilibrium flows that are a small departure from a uniform parallel stream. Its relationship to the classical Prandtl–Glauert equation is apparent. If we set $K = 0$—that is, if the relaxation time τ_∞ is zero—the equation reduces to the Prandtl–Glauert equation for equilibrium flow as one would expect. At the opposite limit of frozen flow (τ_∞ and $K \to \infty$), it goes over into essentially the Prandtl–Glauert equation with the Mach number now based on the frozen rather than the equilibrium speed of sound. For any non-zero value of K, the characteristics of the equation, which are indicative of the character of the solution, are determined by the third-order terms. For example, in two-dimensional flow ($\partial^2\phi/\partial x_3^2 = 0$) there are three characteristics given by the lines y = constant and $dy/dx = \pm 1/\sqrt{(M_{f\infty}^2-1)}$. The existence of the characteristics y = constant, which are the streamlines to the present approximation, is a reflexion of the possibility of differences in entropy from one streamline to the next when non-equilibrium processes are present. The characteristics $dy/dx = \pm 1/\sqrt{(M_{f\infty}^2-1)}$ play essentially the same role as do the corresponding characteristics that are well known in equilibrium flow. They may be real or imaginary depending on whether $M_{f\infty}$ is greater or less than one, and are thus indicative of the hyperbolic or elliptic nature of the flow. This is the situation no matter how small the value of K, so long as it is not zero. When K is identically zero, however, the characteristics change discontinuously, being now determined by the second-order terms. The characteristics y = constant now cease to exist and the remaining characteristics are replaced discontinuously by the lines $dy/dx = \pm 1/\sqrt{(M_{e\infty}^2-1)}$. The equation is therefore now hyperbolic or elliptic depending on whether $M_{e\infty}$ is greater or less than one. We thus see that when K goes from a positive value to zero, the equation will retain a hyperbolic character if both $M_{f\infty}$ and $M_{e\infty}$ are greater than one and an elliptic character if both are less than one. Since $a_{f\infty}$ must always be greater than $a_{e\infty}$, however, there must also exist a range of U_∞ in which $M_{f\infty} < 1$ and $M_{e\infty} > 1$. In this range the equation will change its character discontinuously from elliptic to hyperbolic when $K = 0$. This entire situation is typical of singular perturbation problems as discussed by Lagerstrom, Cole & Trilling (1949). The number of different cases here, however, is greater than one usually finds in a single equation, owing to the possibility of a change in type of both of the linear operators that make up the equation.

Equation (27) for $M_{f\infty} > 1$ is similar to the acoustic equation given by Moore (1958). As pointed out by Moore, an equation of the same general type has also been been derived by Morrison (1956) for the propagation of elastic waves in a material in which there is a lag between stress and strain. For $M_{f\infty} < 1$, equation (27) is also formally equivalent to the equation obtained by Lagerstrom *et al.* in their linearized treatment of stationary waves in a viscous compressible fluid. A third-order equation of this type would thus appear to be representative of a number of small-disturbance processes in which non-equilibrium phenomena are involved.

Equation (27) can be applied to any gas involving a single vibrational or chemical process provided we know $h = h(p, \rho, q)$, $\bar{q} = \bar{q}(p, \rho)$, and τ. For example, in the case of vibrational non-equilibrium of an otherwise perfect diatomic gas, where q is to be identified with the vibrational temperature T_i, the enthalpy is given, in view of the state equation $p/\rho = RT$, by

$$h = \tfrac{7}{2}RT + \int c_i dT_i = \frac{7}{2}\frac{p}{\rho} + \int c_i\, dT_i,$$

where c_i is the vibrational specific heat. The equilibrium value of $\bar{q}$ is given by $\bar{T}_i = T = p/R\rho$. The relaxation time is defined by the linearized rate equation $DT_i/Dt = (T - T_i)/\tau$ and would usually be taken from experimental measurements as a function of T and ρ. Substitution of these quantities into equations (23) and (24) gives

$$a^2_{f\infty} = \tfrac{7}{5}RT_\infty \quad \text{and} \quad a^2_{e\infty} = \frac{\tfrac{7}{2}R + c_{i\infty}}{\tfrac{5}{2}R + c_{i\infty}} RT_\infty,$$

while equation (26) gives

$$K = \frac{\tfrac{7}{2}RU_\infty \tau_\infty}{\tfrac{7}{2}R + c_{i\infty}}.$$

Owing to the neglect of non-linear effects, equation (27) may be expected to suffer from the same shortcomings as does the Prandtl–Glauert equation—that is, it will provide a relatively poor approximation at transonic and hypersonic speeds. As will be seen, however, the discontinuous behaviour that obviously invalidates the solutions of the Prandtl–Glauert equation at transonic speeds disappears with the inclusion of the non-equilibrium process.

Equation (27), like the Prandtl–Glauert equation, could be used as the basis for a theory of thin airfoils. Here, however, we shall examine only the original problem of the wavy wall.

4. Solution for wavy wall

We consider the two-dimensional flow in the half plane above the infinite sinusoidal wall

$$X_2 = \epsilon \sin 2\pi \frac{x_1}{l}, \tag{28}$$

where ϵ denotes the amplitude of the waves and l their wavelength. The boundary condition at the wall is taken in the usual linearized form

$$\left(\frac{\partial \phi}{\partial x_2}\right)_{x_2=0} = u'_2(x_1, 0) = U_\infty \frac{dX_2}{dx_1} = 2\pi U_\infty \frac{\epsilon}{l} \cos 2\pi \frac{x_1}{l}. \tag{29}$$

The boundary conditions at infinity are that $u'_1 = \partial\phi/\partial x_1$ and $u'_2 = \partial\phi/\partial x_2$ remain finite as $x_2 \to \infty$. Since the boundary conditions are unchanged by shifting the origin an integral number of wavelengths in the x_1-direction, it follows at once that ϕ must be periodic in x_1 with period l.

It is convenient to begin by transforming to the new variables $x = 2\pi x_1/l$ and $y = 2\pi x_2/l$. With this transformation and the notation $k = 2\pi K/l$, $a = 1 - M^2_{f\infty}$,

$b = 1 - M_{e\infty}^2$, the differential equation (27) as applied to two-dimensional flow can be written

$$ka\frac{\partial^3\phi}{\partial x^3} + k\frac{\partial^3\phi}{\partial x\,\partial y^2} + b\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0, \tag{30}$$

and the boundary condition (29) becomes

$$\left(\frac{\partial\phi}{\partial y}\right)_{y=0} = U_\infty \epsilon \cos x. \tag{31}$$

The new parameter k in equation (30) is a dimensionless measure of the importance of the non-equilibrium process (cf. equation (26)).

Equation (30) is a linear equation with constant coefficients. One possible solution is given by the exponential form

$$\phi(x, y) = e^{\alpha x - \beta y}, \tag{32}$$

provided α and β are related by the following equation found by substituting the expression (32) into the differential equation (30):

$$\beta^2 = -\alpha^2\frac{b + ka\alpha}{1 + k\alpha}. \tag{33}$$

Here α and β are, in general, complex numbers. The negative sign is taken with β in anticipation of the boundary condition at infinity.

The fact that the solution must be real and periodic suggests that it be taken in the form

$$\phi(x, y) = A\,e^{\alpha x - \beta y} + B\,e^{\alpha^* x - \beta^* y}, \tag{34}$$

where A and B are complex constants and α^* and β^* are the complex conjugates of α and β. In particular, since ϕ is to be periodic in x, we take $\alpha = i$ and $\alpha^* = -i$. With the notation $\beta = \delta + i\lambda$ and $\beta^* = \delta - i\lambda$, where δ and λ are real quantities, the solution (34) can then be written in the purely real form

$$\phi(x, y) = e^{-\delta y}[C\cos(x - \lambda y) + D\sin(x - \lambda y)], \tag{35}$$

where C and D are real constants related to A and B by $C = A + B$ and $D = i(A - B)$.

To find δ and λ, we return to equation (33) and substitute $\beta = \delta + i\lambda$ and $\alpha = i$. (The same result would be obtained by use of the complex conjugates.) By equating the real and imaginary parts of the resulting equation, we find the following simultaneous equations for δ and λ:

$$\delta^2 - \lambda^2 = \frac{b + k^2 a}{1 + k^2} = P\,(\text{say}), \tag{36a}$$

$$2\delta\lambda = \frac{k(a - b)}{1 + k^2} = Q\,(\text{say}). \tag{36b}$$

Note that P can be positive or negative depending on the values of a and b. Q will always be positive, however, since $a_{f\infty} > a_{e\infty}$ with the result that $M_{f\infty} < M_{e\infty}$ and $a > b$. The formal solution of equations (36) is given by

$$\delta = \pm[\tfrac{1}{2}(P \pm \sqrt{[P^2 + Q^2]})]^{\frac{1}{2}}, \quad \lambda = \pm[\tfrac{1}{2}(-P \pm \sqrt{[P^2 + Q^2]})]^{\frac{1}{2}}.$$

Obviously, the absolute value of the radical in these expressions will always be greater than P. The plus sign must therefore be used with the radical to insure that δ and λ will be real. To satisfy the boundary condition at infinity, the sign of δ itself must also be taken as plus. Since Q is always positive in equation (36*b*), it then follows that λ likewise must be plus. With this choice of signs and substitution of P and Q from equations (36), δ and λ are given finally by

$$\left.\begin{matrix}\delta\\ \lambda\end{matrix}\right\} = \left\{\frac{1}{2(1+k^2)}\left[\pm(b+k^2a)+\sqrt{\{(1+k^2)(b^2+k^2a^2)\}}\right]\right\}^{\frac{1}{2}}, \tag{37}$$

where the upper sign goes with δ and the lower with λ.

It remains to satisfy the boundary condition at the wall and thus determine the constants C and D in solution (35). To this end equation (35) is substituted into condition (31) to obtain the following simultaneous equations:

$$\lambda C - \delta D = 0,$$

$$\delta C + \lambda D = -U_\infty \epsilon.$$

These have the solution

$$C = -\frac{\delta}{\delta^2+\lambda^2}U_\infty\epsilon, \quad D = -\frac{\lambda}{\delta^2+\lambda^2}U_\infty\epsilon.$$

The potential (35) can thus be written finally as

$$\phi(x,y) = -\frac{U_\infty \epsilon}{\delta^2+\lambda^2}e^{-\delta y}\left[\delta\cos(x-\lambda y)+\lambda\sin(x-\lambda y)\right], \tag{38}$$

where δ and λ are given by equation (37). In obtaining this solution no distinction has been necessary between subsonic and supersonic flow.

5. Discussion of solution

A. *Flow field*

The potential function (38), when expressed in terms of the original variables x_1 and x_2, yields the following expressions for the disturbance velocies:

$$\frac{u_1'}{U_\infty} = \frac{1}{U_\infty}\frac{\partial\phi}{\partial x_1} = \frac{2\pi(\epsilon/l)}{\delta^2+\lambda^2}e^{-2\pi\delta(x_2/l)}\left(\delta\sin 2\pi\frac{x_1-\lambda x_2}{l} - \lambda\cos 2\pi\frac{x_1-\lambda x_2}{l}\right), \tag{39}$$

$$\frac{u_2'}{U_\infty} = \frac{1}{U_\infty}\frac{\partial\phi}{\partial x_2} = 2\pi\frac{\epsilon}{l}e^{-2\pi\delta(x_2/l)}\cos 2\pi\frac{x_1-\lambda x_2}{l}. \tag{40}$$

The horizontal disturbance velocity can also be written

$$\frac{u_1'}{U_\infty} = \frac{2\pi(\epsilon/l)}{\sqrt{(\delta^2+\lambda^2)}}e^{-2\pi\delta(x_2/l)}\sin 2\pi\frac{(x_1-\Delta x_1)-\lambda x_2}{l}, \tag{39a}$$

where $\Delta x_1/l = (1/2\pi)\tan^{-1}(\lambda/\delta)$.

Before examining the detailed nature of the functions δ and λ, some general results can be stated. Under all conditions the disturbance velocities decay exponentially with x_2 along the lines $x_1 - \lambda x_2 =$ constant, which are straight lines with slope (measured from the vertical)

$$\frac{dx_1}{dx_2} = \lambda. \tag{41}$$

The rate of decay is proportional to the value of δ. Along the straight lines (41) the vertical disturbance velocity is in phase with the slope of the wall. The horizontal disturbance velocity, however, is out of phase with the wall. For a given value of x_2, it lags behind the ordinate of the wall by the horizontal distance Δx_1 as given in equation (39*a*).

To see what the foregoing results mean in terms of Mach number, we must examine the dependence of δ and λ on $M_{f\infty}$ and $M_{e\infty}$. In terms of these variables, equation (37) gives

$$\left.\begin{matrix}\delta\\ \lambda\end{matrix}\right\} = \left[\frac{1}{2(1+k^2)}\{\pm(1-M_{e\infty}^2) \pm k^2(1-M_{f\infty}^2) + \sqrt{[(1+k^2)\{(1-M_{e\infty}^2)^2 + k^2(1-M_{f\infty}^2)^2\}]}\}\right]^{\frac{1}{2}} \tag{42}$$

For the limiting cases of $k = 0$ (equilibrium flow) and $k = \infty$ (frozen flow), equation (42) reduces to the simple results of the following table:

	$k = 0$. Equilibrium flow		$k = \infty$. Frozen flow	
	Subsonic $M_{e\infty} < 1$	Supersonic $M_{e\infty} > 1$	Subsonic $M_{f\infty} < 1$	Supersonic $M_{f\infty} > 1$
δ	$\sqrt{(1-M_{e\infty}^2)}$	0	$\sqrt{(1-M_{f\infty}^2)}$	0
λ	0	$\sqrt{(M_{e\infty}^2-1)}$	0	$\sqrt{(M_{f\infty}^2-1)}$

In both limiting cases therefore, the disturbance velocities are given by equations of the following form:

For subsonic flow,

$$\frac{u_1'}{U_\infty} = \frac{2\pi(\epsilon/l)}{\sqrt{(1-M_\infty^2)}}\exp\left[-2\pi\sqrt{(1-M_\infty^2)}\frac{x_2}{l}\right]\sin 2\pi\frac{x_1}{l},$$

$$\frac{u_2'}{U_\infty} = 2\pi\frac{\epsilon}{l}\exp\left[-2\pi\sqrt{(1-M_\infty^2)}\frac{x_2}{l}\right]\cos 2\pi\frac{x_1}{l}.$$

For supersonic flow,

$$\frac{u_1'}{U_\infty} = -\frac{2\pi(\epsilon/l)}{\sqrt{(M_\infty^2-1)}}\cos 2\pi\frac{x_1-\sqrt{(M_\infty^2-1)}\,x_2}{l},$$

$$\frac{u_2'}{U_\infty} = 2\pi\frac{\epsilon}{l}\cos 2\pi\frac{x_1-\sqrt{(M_\infty^2-1)}\,x_2}{l}.$$

These are of the same form as the classical results obtained by Ackeret on the basis of the Prandtl–Glauert equation (see, for example, Liepmann & Roshko (1957)). The decay of the velocity field is zero for supersonic flow, and the rearward rotation of the lines along which the decay takes place is zero for subsonic flow. As one would expect, the only difference between the results for equilibrium and frozen flow is in the speed of sound on which the Mach number is based.

The variation of δ and λ for an intermediate value of k is shown in figure 1, together with curves for the two limiting cases. The results are plotted as functions of $M_{e\infty}$ for a value of the ratio $a_{f\infty}/a_{e\infty}$ of 11/10, which is a convenient value, though perhaps slightly larger than one would actually find for a single non-equilibrium process in a diatomic gas. It follows that $M_{f\infty} = (a_{e\infty}/a_{f\infty}) M_{e\infty} = (10/11) M_{e\infty}$. The intermediate value of $k = 1$ was chosen as giving very nearly the maximum value of the variables in the regions in which the variation with k is not monotonic.

Figure 1 shows that the existence of a finite relaxation time removes certain of the qualitative differences that distinguish subsonic flow from supersonic flow in the two limiting cases. (The transition from one type of flow to the other, in fact, is no longer clearly defined.) The exponential decay of the velocity field, as measured by the value of δ, now persists throughout the Mach-number range. As before, it decreases rapidly in the near-sonic region. It never disappears entirely,

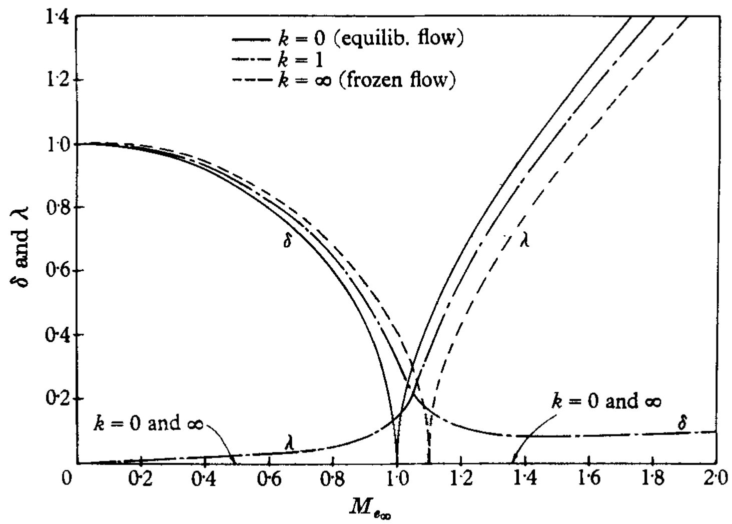

FIGURE 1. Values of δ and λ as functions of $M_{e\infty}$ for $a_{f\infty}/a_{e\infty} = 11/10$.

however, as it does for both equilibrium and frozen flow at supersonic speed. The rearward rotation of the lines of exponential decay, which is measured by the value of λ, is now zero for incompressible flow only ($M_{e\infty} = 0$). As in the two limiting cases, it grows rapidly in the near-sonic region. It is evident to a slight extent, however, even at lower speeds.

Other things being equal, the effect of differences in relaxation time is shown by the variation of δ and λ with k for fixed $M_{e\infty}$. For a value of $M_{e\infty} < 1$ (see figure 1), δ increases monotonically with increasing k, whereas λ rises to a maximum and then declines. In the limits of $k = 0$ and ∞ the disturbance field is of the classical subsonic type with marked exponential decay along vertical lines. For a value of $M_{e\infty} > a_{f\infty}/a_{e\infty}$, the situation is reversed: λ varies monotonically and δ exhibits a maximum. In the two limiting cases the field is now of the classical linearized supersonic type with the velocities propagating undiminished along rearward sloping lines. For $1 < M_{e\infty} < a_{f\infty}/a_{e\infty}$, the variation of δ and λ with k is rather complicated. In the limit of $k = 0$ (equilibrium flow), the disturbance field is of the classical supersonic type; for $k = \infty$ (frozen flow), it is of the classical subsonic type.

The entire situation with regard to variations in both $M_{e\infty}$ and k is exemplified in figure 2, which shows typical streamline patterns for three values each of both of these parameters. The behaviour observed here is related, of course, to the various possibilities as regards the elliptic or hyperbolic character of the two parts of the differential equation as previously discussed.

It is worth noting that the classical supersonic flow with propagation along outgoing (i.e. rearward sloping) Mach lines appears here as the natural limit of the non-equilibrium solution. In the classical analysis, which sets $k = 0$ in the

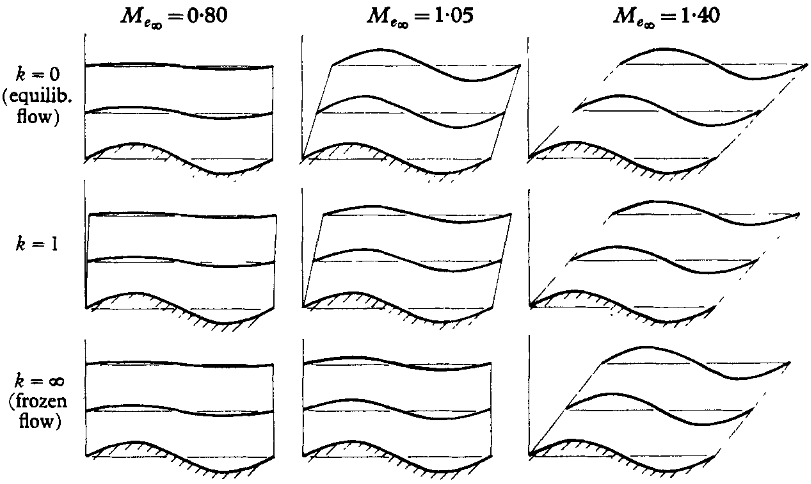

FIGURE 2. Typical streamline patterns for $\epsilon/l = \frac{1}{12}$.

differential equation at the outset, there exists in addition a mathematical solution representing propagation along the incoming Mach lines, and this solution must be discarded on physical grounds. In the present analysis, the non-equilibrium (i.e. irreversible) process determines a unique direction of propagation, and this automatically picks out the physically correct result in the limit.

B. *Pressure distribution and drag coefficient*

To the accuracy required in the two-dimensional small-disturbance case, the pressure coefficient $C_p \equiv (p-p_\infty)/\frac{1}{2}\rho_\infty U_\infty^2$ can be found from equation (13) as

$$C_p = -2\frac{u_1'}{U_\infty}.$$

Substitution from equation (39*a*) gives for the pressure coefficient on the wall

$$(C_p)_{x_2=0} = -\frac{4\pi(\epsilon/l)}{\sqrt{(\delta^2+\lambda^2)}}\sin 2\pi\frac{x_1-\Delta x_1}{l}. \tag{43}$$

A plot of $\Delta x_1/l = (1/2\pi)\tan^{-1}(\lambda/\delta)$ for the conditions of figure 1 is shown in figure 3. For $k = 1$ the point of minimum pressure on the wall is seen to shift continuously backward from the crest toward the point of maximum negative slope as the Mach number increases. This is in contrast to the situation in the Ackeret solution ($k = 0$ and ∞), where a shift of one-quarter wavelength takes place discontinuously as the flow changes from subsonic to supersonic. The results of equation (43) and figure 1 also show that the pressure coefficient in non-equilibrium flow always remains finite. This is again in contrast to both limiting cases, where C_p becomes indeterminately large at the sonic speed.

The drag coefficient per wavelength of wall can be calculated from

$$c_d = \int_0^1 (C_p)_{x_2=0} \frac{dX_2}{dx_1} d\left(\frac{x_1}{l}\right).$$

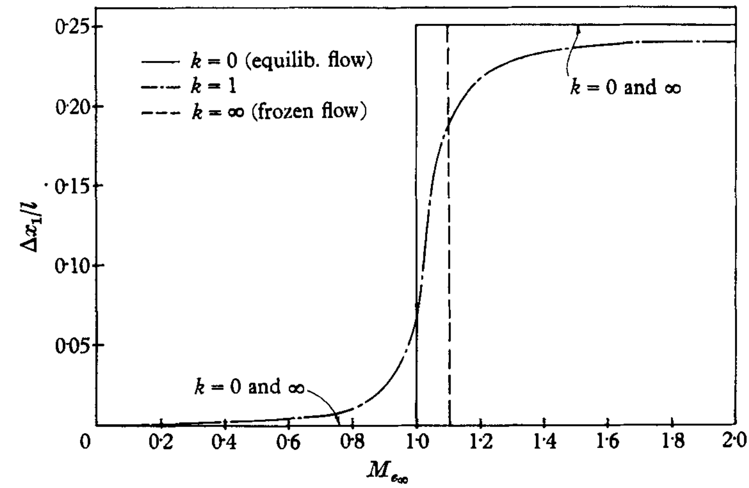

FIGURE 3. Relative shift of pressure distribution as a function of $M_{e\infty}$ for $a_{f\infty}/a_{e\infty} = \frac{11}{10}$.

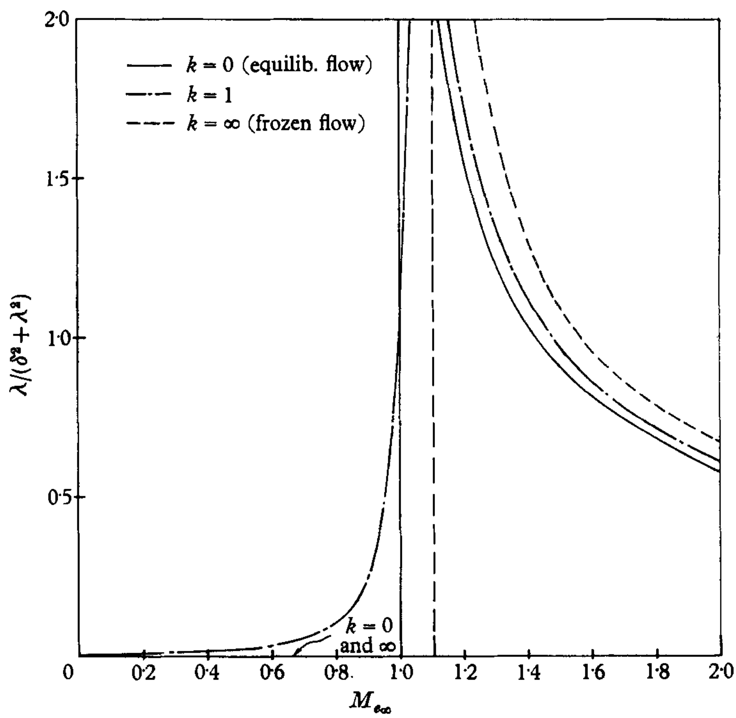

FIGURE 4. Drag parameter as a function of $M_{e\infty}$ for $a_{f\infty}/a_{e\infty} = \frac{11}{10}$.

Substitution from equations (28) and (43) gives

$$c_d = 4\pi^2 \left(\frac{\epsilon}{l}\right)^2 \frac{\lambda}{\delta^2+\lambda^2}. \tag{44}$$

The factor $\lambda/(\delta^2+\lambda^2)$ is plotted as a function of $M_{e\infty}$ in figure 4. As would be expected from the previous results, the non-equilibrium effects eliminate the discontinuous behaviour in drag coefficient that is a characteristic feature of the classical Ackeret solution. The maximum in the near-sonic range is still sharp—so much so that the labour required to find it precisely did not seem worthwhile. It is, nevertheless, not infinite. The non-zero drag that is now evident for $M_{e\infty} < 1$ is due to the presence of a mechanism for entropy increase even at subsonic speeds (cf. equation (11)). At supersonic speeds there are now two sources of entropy rise: shock waves, whose action is well known from classical gas dynamics, and non-equilibrium effects, which now act in the regions between shock waves. The net effect is apparently to cause the drag for $M_{e\infty} > a_{f\infty}/a_{e\infty}$ to lie between the limiting values calculated for equilibrium and frozen flow.

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